many instances a factor of 1.7 speedup was obtained. The solver is now being extended to three dimensions.

Acknowledgments

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Fast Three-Dimensional Vortex Method for Unsteady Wake Calculations

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Introduction

RECENTLY, there has been much interest in the use of a Lagrangian vortex method for the simulation of unsteady vortical flows.[1] In this method, the moving packets of vorticity are discretized into a collection of vortex elements. The elements convect at the local velocity and automatically track the vorticity packets as they evolve in the flow. The velocity is computed from a direct summation of the Biot-Savart induction, and for N vortex elements, the calculation has an asymptotic time complexity of $O(N^2)$.

Results and Discussion

In the following, we describe a fast vortex method with enhanced computational efficiency. In this method, a grid of three-dimensional cubic boxes is superimposed on the computational domain, and the elements that reside in each box are clustered into a group. The velocity at a given observation point is computed from two components: a near-field vortex-induced velocity and a far-field group-induced velocity. The near-field velocity is computed by summing the Biot-Savart velocity due to all vortices that reside in the same box or immediate-neighboring boxes as the observation point. Vortices that reside in boxes that are well-separated from that of the observation point, i.e., more than one box length away, are considered to be in the far field, and their velocity inductions are computed using multipole expansions and Taylor series expansions.

The Biot-Savart velocity induction due to a group of vortex elements is given by

$$u(x,t) = -\frac{1}{4\pi}\sum_{i=1}^{N}\frac{[x-x_i(t)]\times\omega_i(t)\,\delta v}{|x-x_i(t)|^3} \tag{1}$$

where x_i, ω_i, and δv are the position, vorticity strength, and vorticity volume, respectively, of the ith vortex element. A multipole expansion of Eq. (1) can be written as

$$u(x) = -\frac{1}{4\pi r^3}\sum_{k=0}^{\infty}E_k\frac{1}{r^k} \tag{2}$$

where E_k are the moment coefficients of the expansion $r = x - x_{cm}$, and x_{cm} is the point of expansion. Provided that $r > D$, where D is the group diameter, the multipole expansion converges. In particular, the truncation error of the expansion is bounded by $c(D/r)^{p+1}$, where p is the number of terms in the expansion and c is a constant.[2] Using the multipole expansion, the computational speed of the vortex method can be accelerated. This acceleration is because the inducting effects of a large number of vortices are replaced by a single group induction computed using a small number of numerical terms.

A further improvement in the computational speed can be attained using a Taylor series expansion. Here, the far-field group-induced velocity is evaluated at the centroid x_{cm} of a group of observation points, and a Taylor series expansion is used to compute the velocity at individual points within the group:

$$u(x) = u(x_{cm}) + (x-x_{cm})\cdot\nabla u(x_{cm}) + \frac{1}{2!}[(x-x_{cm})\cdot\nabla]^2u(x_{cm}) + \ldots \tag{3}$$

In the following, the accuracy of the multipole expansions is examined. The test problem involves a single thin vortex ring on the x-y plane. The ring has a radius $R = 1$ and a circulation $\Gamma = 4\pi$. It is represented as a single filament discretized into an array of 500 vortex elements. All of the elements are clustered into a group and the group-induced velocity is computed, using multipole expansions, at a set of observation points placed along an axial line intercepting the vortex ring. Three different cases with one, two, and three terms in the expansion are computed, and the velocities are compared to that obtained using a direct velocity summation. Figure 1 shows a plot of the relative velocity error $(U_a - U_e)/U_e$, where U_a is computed using a multipole expansion, and U_e is computed using a direct summation. A schematic illustrating the problem geometry is included as an insert in the figure. For points close to the vortex ring, all three cases give considerable error. This error is because the ratio $D/z > 1$, where $D = 2$, is the diameter of the vortex group, and the multipole expansion fails to converge. For points more than one diameter away from the ring, the error falls off rapidly as the number of multipole terms is increased.

The propagation of the vortex ring is computed using the fast vortex method, and the computed propagation speed is compared

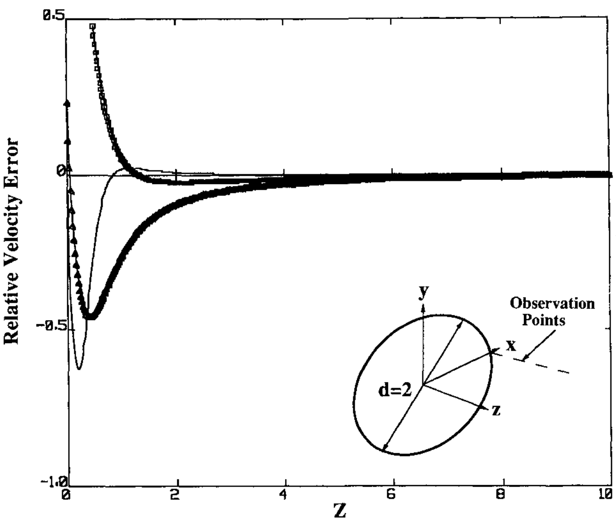

Fig. 1 Comparison of relative velocity error for multipole expansions, with one term—□; two terms—Δ; and three terms————, computed for the vortex ring problem (insert).

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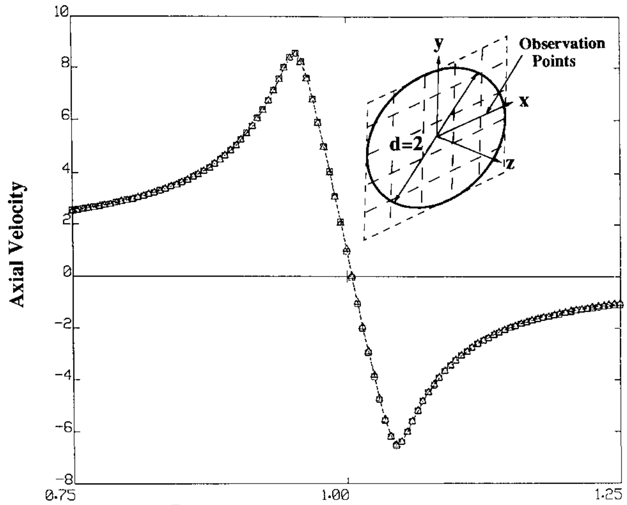

Fig. 2 Comparison of vortex-induced velocity, computed using fast vortex method—Δ and direct summation—□, at points along the x axis for the vortex ring problem (insert).

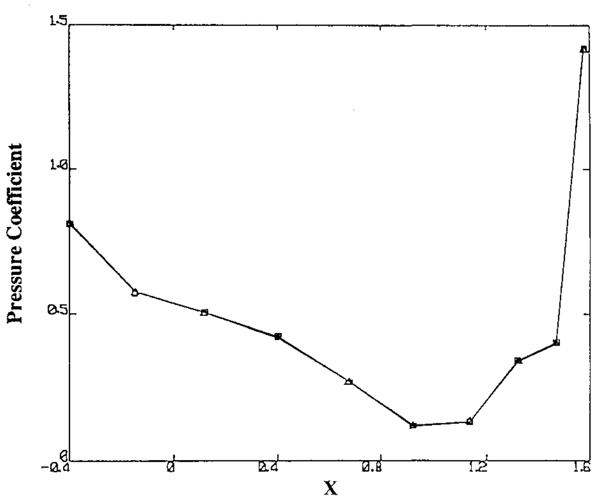

Fig. 3 Comparison of pressure coefficient on an axisymmetric fuselage immersed in a rotor wake computed using fast method—Δ and direct summation—□.

with an analytical solution.[1] The vortex ring, as described earlier, is represented as a single filament discretized using 100 elements. Each element has a smooth Rosenhead core $\sigma = 0.05R$. This corresponds to a thick vortex ring with a cross-sectional thickness of $0.05R$. A uniform grid of 10×10 cubic boxes, each with a dimension of $0.2R$, is superimposed on the domain to completely contain the vortex ring. The elements are clustered into groups and their induced velocity computed using the fast vortex method. Figure 2 shows a plot of the induced axial velocity computed at points along the x axis. Data points computed using the fast vortex method are shown as triangles. Three terms are used in the multipole expansion and two terms in the Taylor series expansion. They are compared with a calculation using a direct velocity summation (represented as squares). Good agreement is observed. The unsteady motion of the vortex ring is computed using a fourth-order Runge-Kutta scheme with a time step of 0.01. In the calculation, the grid of boxes is convected with the ring and the vortex-induced velocity is computed using the fast vortex method. From the simulation, the vortex ring propagation speed is computed to be 0.84386 and agrees very well with the analytical speed of 0.84456.

In the following, we describe an application of the fast vortex method to an unsteady rotor/fuselage wake problem. The flow involves a complex three-dimensional wake trailing from a four-bladed rotor and draping over an axisymmetric fuselage. The wake is modeled using vortex filaments, and the fuselage is modeled using source panels. A detailed description of the analysis program

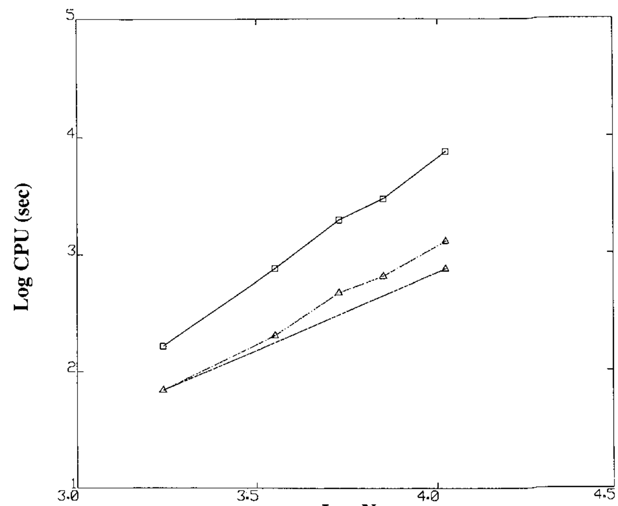

Fig. 4 Comparison of computation time(s) for the velocity calculation, using fast vortex method - Δ and direct summation—□; theoretical limit for fast vortex method————.

is given in Ref. 3. In the calculations, the wake trailed from each rotor blade is represented using 12 filaments, each discretized with 72–160 vortex elements, and the axisymmetric fuselage is represented using a grid of 35×16 source panels. Because of the large number of elements used in the analysis, the flow problem is computationally intensive and is well suited to an application of the fast vortex method.

A grid of $15 \times 6 \times 2$ cubic boxes is superimposed on the flow domain that contains the four-bladed rotor, the body of revolution representing the fuselage, and the vortical wake. Vortex elements residing in each box are clustered into a group, and the group inductions are used in computing the far-field velocity. The calculations are carried out over several rotor revolutions, and the result is compared with that obtained using a direct velocity summation. Figure 3 shows a comparison of the pressure coefficient at a set of points along the symmetry plane on the top surface of the fuselage. The triangles are obtained from the fast vortex calculations, and the squares are obtained from the direct calculations. Very good agreement between the two calculations is observed. Figure 4 shows a comparison of the velocity calculation time (in seconds) per step using the fast vortex method and using a direct summation. The computations have been carried out on a Silicon Graphics Iris 4D/220S workstation. In the calculations using a direct summation (represented as squares), the logarithmic plot shows a slope of 2, which confirms the N^2-dependence of the computation time. In the fast vortex calculations (represented as triangles), the computation time is estimated to be $\sim N^{1.63}$. For comparison, the ideal limit of $N^{4/3}$ is also plotted. This limit is attained when the calculation is optimized, i.e., the number of near-field vortex-to-vortex calculations and the number of far-field group-to-group calculations are comparable.

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